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On fuzzy spheres and (M)atrix actions

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Abstract

In this note we compare even and odd fuzzy sphere constructions, their dimensional reductions and possible (M)atrix actions having them as solutions. We speculate on how the fuzzy 5-sphere might appear as a solution to the pp wave (M)atrix model.

Fuzzy spheres are of physical relevance (beyond their interesting mathematical aspects) because of the possibility that they appear as solutions to (M)atrix theory. As such, they give a possible quantum version of classical sphere solution, one which might be of relevance in the early Universe physics (see e.g. [1]), as well as providing possible vacuum solutions in certain backgrounds. The fuzzy S^2 in fact is a solution to the M theory Matrix model in the maximally supersymmetric pp wave background [2].

We will therefore review some of the aspects of fuzzy spheres constructions, simplifying the analysis of odd spheres. We will see that in the described context, dimensional reduction of spheres becomes easier to understand (even if still not straightforward). We will write down Matrix model actions that have the even and odd spheres as solutions and argue that such an action will probably describe the quantum corrected version of the pp wave (M)atrix model, thus giving the conjectured 5-brane solution [2].

Fuzzy S^2

The best understood case of fuzzy sphere is the fuzzy S^2. One needs noncommutative coordinates (realized by infinite matrices in the (M)atrix theory case) that satisfy an SO(3)-invariant algebra, generalizing the classical sphere. The algebra is

$$[X^i, X^j] = iR\epsilon^{ijk}X^k; \quad (X^i)^2 = R^2 \tag{1}$$

By multiplying with ϵ^{ijk}, the defining algebra becomes

$$\epsilon^{ijk}X^iX^j = iRX^k \tag{2}$$

which is equivalent to the previous.

Fuzzy S^4

The case of fuzzy S^4 was analyzed in [3] following earlier work in [4, 5]. Given the knowledge that it had to be a solution to the (M)atrix action carrying 4-brane charges, the authors defined the algebra to be satisfied as (again, a suitable SO(5)-invariant extension of the classical 4-sphere)

$$\begin{aligned}
&\epsilon^{ijklm}X_iX_jX_kX_l = \alpha X^m \\
&(X^i)^2 = R^2 \\
&R_{ij}X_j = U(R)X_iU(R^{-1})
\end{aligned} \tag{3}$$

It is not clear that this definition is equivalent to

$$[X_i, X_j] = \beta\epsilon_{ijklm}X^kX^lX^m \tag{4}$$

which would be another possible definition of the fuzzy S^4. As we noted, in the S^2 case the two definitions are identical.

The explicit construction of the fuzzy S^4 though does not cover all possible 4-brane charges, only those that can be written as

$$N = \frac{(n+1)(n+2)(n+3)}{6} \tag{5}$$

In that case, the construction is

$$X^i = \sum_r \rho_r(\Gamma^i); \quad X^i : (V^{\otimes n})_{sym} \to (V^{\otimes n})_{sym} \tag{6}$$

where $\rho_r(X)$ inserts X on the r position in $V^{\otimes n}$, and V is the vector space of spinor representation for SO(5). This explicit construction satisfies both (3) and (4).

The explicit construction of the fuzzy S^4 also satisfies the equations of motion

$$[X^j, [X^i, X^j]] + aX^i = 0 \tag{7}$$

since $J^{ij} = [X^i, X^j] = \sum_r \rho_r([\Gamma_i, \Gamma_j])$ are the generators of SO(5). But it is not clear if these equations of motion are a consequence of one of the original forms of the algebra (3) or (4). The equations of motion in (7) come from the action

$$S = \int \frac{[X^i, X^j]^2}{2} + a(X^i)^2 \tag{8}$$

For S^2, these equations of motion can also be rewritten as

$$[J_{ij}, X_j] = 4X_i \tag{9}$$

since $J_{ij} \propto [X_i, X_j]$, except that now one can easily see that the fuzzy S^2 algebra implies these equations of motion.

Fuzzy S^{2k}

The construction generalizes easily to even spheres S^{2k} as (the detailed analysis was done in [6, 7] and further clarified in [8, 9])

$$\begin{aligned} &\epsilon^{i_1 \dots i_{2k}} X_{i_1} \dots X_{i_{2k-1}} = \alpha X^{i_{2k}} \\ &(X^i)^2 = R^2 \\ &R_{ij} X_j = U(R) X_i U(R^{-1}) \end{aligned} \tag{10}$$

or (equivalently?)

$$[X_i, X_j] = \beta \epsilon_{iji_2 \dots i_{2k}} X^{i_3} \dots X^{i_{2k}} \tag{11}$$

solved by the explicit construction $X_i = \sum_r \rho_r(\Gamma_i)$ that satisfies the equations of motion

$$[J_{ij}, X_j] = 4kX_i \tag{12}$$

due to $J_{ij} \propto [X_i, X_j]$.

Odd dimensional spheres: construction and algebra

We are again looking for a matrix representation that generalizes the coordinates X^i, and since we want a generalization of a sphere, we want to have $(X^i)^2 = $ const. and an algebra (coming hopefully from some simple equations of motion) which are SO(2n)-invariant. The fuzzy 3-sphere was introduced in [10] and further developped and generalized to odd spheres in [11].

We will follow the analysis in [11], and we will focus on the S^3 case, leaving the generalization to any odd spheres to the end.

Take the vector space V of spinor representations for SO(4). It splits into positive and negative chirality representations, as $SO(4) = SU(2) \times SU(2)$, thus $V = V_+ + V_-$. Take the subspace $\mathcal{R}_n$ of $Sym(V^{\otimes n})$. It will substitute $Sym(V^{\otimes n})$ as the basis vector space for the fuzzy sphere representation. $\mathcal{R}_n$ is defined as $\mathcal{R}_n^+ + \mathcal{R}_n^-$, where $\mathcal{R}_n^+$ and $\mathcal{R}_n^-$ are spaces of $SU(2) \times SU(2)$ labels $(\frac{n+1}{2}, \frac{n-1}{2})$ and $(\frac{n-1}{2}, \frac{n+1}{2})$ respectively (denoting the number of V_+ and V_- factors in each).

Then

$$X^i = \mathcal{P}_{\mathcal{R}_n} \hat{X}_i \mathcal{P}_{\mathcal{R}_n} = \mathcal{P}_{\mathcal{R}_n^+} \hat{X}_i \mathcal{P}_{\mathcal{R}_n^-} + \mathcal{P}_{\mathcal{R}_n^-} \hat{X}_i \mathcal{P}_{\mathcal{R}_n^+} = X_i^+ + X_i^- \tag{13}$$

where $\mathcal{P}_{\mathcal{R}_n} = \mathcal{P}_{\mathcal{R}_n^+} + \mathcal{P}_{\mathcal{R}_n^-}$ is the projector and

$$\hat{X}_i = \sum_r \rho_r(\Gamma_i) \tag{14}$$

Then $X_i = X_i^+ + X_i^-$ and $Y^i = X_i^+ - X_i^-$ are two independent variables that need to be used to define the fuzzy S^3, or equivalently we can use X_i^+, X_i^-.

Following [11], we can compute that X_i^2 commutes with the SO(4) generators and is indeed constant in this subspace,

$$X_i^2 \mathcal{P}_{\mathcal{R}_n} = \frac{(n+1)(n+3)}{2} \equiv N \tag{15}$$

For a fuzzy S^{2k-1}, one gets $(n+1)(n+2k-1)/2$ on the r.h.s.

One needs to take the irrep $\mathcal{R}_n$ as opposed to $Sym(V^{\otimes n})$ in the even sphere case, and then X_i^2 is constant on the space. In the calculation, one needs to be careful, since

$$\sum_i \mathcal{P}_{\mathcal{R}_n} \sum_r \rho_r(\Gamma_i) \sum_s \rho_r(\Gamma_i) \mathcal{P}_{\mathcal{R}_n} \neq \sum_i X_i^2 \tag{16}$$

For the definition of the fuzzy 3-sphere algebra, [11] defines the objects

$$\begin{aligned}
X_i^+ &= \mathcal{P}_{\mathcal{R}_n^-} \sum_r \rho_r(\Gamma_i P_+) \mathcal{P}_{\mathcal{R}_n^+} & X_i^- &= \mathcal{P}_{\mathcal{R}_n^+} \sum_r \rho_r(\Gamma_i P_-) \mathcal{P}_{\mathcal{R}_n^-} \\
X_{ij}^+ &= \mathcal{P}_{\mathcal{R}_n^+} \sum_r \rho_r(\frac{1}{2}[\Gamma_i, \Gamma_j] P_+) \mathcal{P}_{\mathcal{R}_n^+} & Y_{ij}^+ &= \mathcal{P}_{\mathcal{R}_n^+} \sum_r \rho_r(\frac{1}{2}[\Gamma_i, \Gamma_j] P_-) \mathcal{P}_{\mathcal{R}_n^+} \\
X_{ij}^- &= \mathcal{P}_{\mathcal{R}_n^-} \sum_r \rho_r(\frac{1}{2}[\Gamma_i, \Gamma_j] P_-) \mathcal{P}_{\mathcal{R}_n^-} & Y_{ij}^- &= \mathcal{P}_{\mathcal{R}_n^-} \sum_r \rho_r(\frac{1}{2}[\Gamma_i, \Gamma_j] P_+) \mathcal{P}_{\mathcal{R}_n^-}
\end{aligned} \tag{17}$$

where $P_\pm$ are projectors onto $V_\pm$ and then also

$$\begin{aligned} X_i &= X_i^+ + X_i^- \qquad & Y_i &= X_i^+ - X_i^- \\ X_{ij} &= X_{ij}^+ + X_{ij}^- \qquad & Y_{ij} &= Y_{ij}^+ + Y_{ij}^- \\ \tilde{X}_{ij} &= X_{ij}^+ - X_{ij}^- \qquad & \tilde{Y}_{ij} &= Y_{ij}^+ - Y_{ij}^- \end{aligned} \tag{18}$$

where all X_{ij}^+ is selfdual ($X_{ij}^+ = 1/2\epsilon_{ijkl}X_{kl}^+$) and so is Y_{ij}^-, whereas X_{ij}^- and Y_{ij}^+ are anti-selfdual. The algebra is then defined by a large series of (anti)commutators between these generators.

However, one can easily check that that algebra implies

$$\begin{aligned} &(n+3)X_{ij} = [X_i, X_j] + \frac{1}{2}\epsilon_{ijkl}\{X_k, Y_l\}; \;\; X_{ij} = \frac{1}{2}\epsilon_{ijkl}\tilde{X}_{kl} \\ &-(n+1)Y_{ij} = [X_i, X_j] - \frac{1}{2}\epsilon_{ijkl}\{X_k, Y_l\}; \;\; Y_{ij} = -\frac{1}{2}\epsilon_{ijkl}\tilde{Y}_{kl} \\ &(n+3)\tilde{X}_{ij} = \{X_i, Y_j\} + \epsilon_{ijkl}X_kX_l \\ &-(n+1)\tilde{Y}_{ij} = \{X_i, Y_j\} - \epsilon_{ijkl}X_kX_l \end{aligned} \tag{19}$$

and thus only X_i and Y_i parametrize the fuzzy algebra, subject to the constraint

$$[X_i, X_j] = -[Y_i, Y_j] \tag{20}$$

Since $\mathcal{P}_{\mathcal{R}_n^-}\mathcal{P}_{\mathcal{R}_n^+} = \mathcal{P}_{\mathcal{R}_n^+}\mathcal{P}_{\mathcal{R}_n^-} = 0$, $X_i^+X_j^- = X_i^-X_j^+ = 0$, and thus $X_i^2 = X_i^+X_i^- + X_i^-X_i^+ = -Y_i^2 = N$.

From the explicit form in (18) one can find that the SO(4) generators are

$$J_{ij} \equiv \mathcal{P}_{\mathcal{R}_n}\sum_r \rho_r(\frac{1}{2}[\Gamma_i, \Gamma_j])\mathcal{P}_{\mathcal{R}_n} = X_{ij} + Y_{ij} \tag{21}$$

and so

$$J_{ij} = -\frac{2}{(n+1)(n+3)}[X_i, X_j] + \frac{(n+2)}{(n+1)(n+3)}\epsilon_{ijkl}\{X_k, Y_l\} \equiv A[X_i, X_j] + B\epsilon_{ijkl}\{X_k, Y_l\} \tag{22}$$

where $A = -1/N$ and $B = (n+2)/(2N) = \sqrt{2N+1}/(2N)$.

With this definition, the only independent equation of motions, giving the algebra of the fuzzy sphere in terms of X_i, Y_i, are the rotation properties of X_i, Y_i, namely

$$[J_{ij}, X_j] = 6X_i; \quad [J_{ij}, Y_j] = 6Y_i \tag{23}$$

where of course one has to replace J_{ij} with its explicit form as a function of X_i, Y_i, and always subject to the contraint $[X_i, X_j] = -[Y_i, Y_j]$. This is quite satisfying, given that the even fuzzy sphere equations were also related to rotational invariance (there, through the presence of the epsilon tensor).

One has to find actions that have these equations of motion. But first notice that Y_i is anti-hermitian, thus we need to redefine $Y_i = i\tilde{Y}_i$ and thus find an action for $X_i, \tilde{Y}_i$ modulo the constraint $[X_i, X_j] = [\tilde{Y}_i, \tilde{Y}_j]$.

Actions

One can easily check that such an action is (one writes such an action with arbitrary coeffficients for all the terms and then fixes the coefficients by requiring to get the correct equations of motion)

$$L = \frac{1}{2N} Tr\{\frac{1}{4}([X_i, X_j]^2 - [\tilde{Y}_i, \tilde{Y}_j]^2) - 3NX_i^2 + 3N\tilde{Y}_i^2 + i\frac{\sqrt{2N+1}}{2}\epsilon_{ijkl}(X_iX_j\{X_k, \tilde{Y}_l\} + \tilde{Y}_i\tilde{Y}_j\{\tilde{Y}_k, X_l\})\} \tag{24}$$

One notes that the $\tilde{Y}$ terms have the wrong sign. Thus the energy of a static solution (so that $E = -L$) satisfying $[X_i, X_j] = [\tilde{Y}_i, \tilde{Y}_j]$ and $X_i^2 = \tilde{Y}_i^2$ (constraints) is easily seen to be zero. The same was true for the fuzzy S^2 solution to the pp wave (M)atrix model in [2], except that in that case we needed the actual equations of motion, not just a constraint.

The construction of the algebra and action for the fuzzy S^5 case is similar. One can still define the objects in (17,18) subject to the constraint that $[X_i, X_j] = -[Y_i, Y_j]$. We will get similar equations of motion (by simplifying the algebra). Without doing the explicit calculations, from SO(6) invariance and using the constraint, we can say that the SO(6) generator is

$$J_{ij} = X_{ij} + Y_{ij} = -a[X_i, X_j] + b\epsilon_{ijklmn}X_kX_l\{X_m, Y_n\} \tag{25}$$

(where a and b could be determined by explicit calculation), with equations of motion

$$[X_{ij} + Y_{ij}, X_i] = 10X_i; \quad [X_{ij} + Y_{ij}, Y_j] = 10Y_j \tag{26}$$

Then the corresponding action is (after redefining $Y_i = i\tilde{Y}_i$)

$$L = Tr\{\frac{a}{4}([X_i, X_j]^2 - [\tilde{Y}_i, \tilde{Y}_j]^2) - 5(X_i^2 - \tilde{Y}_i^2) + i\frac{b}{2}\epsilon_{ijklmn}(X_iX_jX_kX_l\{X_m, \tilde{Y}_n\} + \tilde{Y}_i\tilde{Y}_jX_kX_l\{\tilde{Y}_m, X_n\})\} \tag{27}$$

This construction generalizes also easily to S^{2k-1}. Using similar arguments as for the S^5 case, the generator of $SO(2k)$ rotations is

$$J_{ij} = X_{ij} + Y_{ij} = -a[X_i, X_j] + b\epsilon_{iji_3...i_{2k}}X_{i_3}...X_{i_{2k-2}}\{X_{i_{2k-1}}, Y_{i_{2k}}\} \tag{28}$$

and the equations of motion are

$$[X_{ij} + Y_{ij}, X_j] = 2(2k-1)X_i; \quad [X_{ij} + Y_{ij}, Y_j] = 2(2k-1)Y_j \tag{29}$$

They come from the action

$$L = Tr\{\frac{a}{4}([X_i, X_j]^2 - [\tilde{Y}_i, \tilde{Y}_j]^2) - (2k-1)(X_i^2 - \tilde{Y}_i^2) + i\frac{b}{2}\epsilon_{i_1...i_{2k}}(X_{i_1}...X_{i_{2k-2}}\{X_{i_{2k-1}}, Y_{i_{2k}}\} + Y_{i_1}Y_{i_2}X_{i_3}...X_{i_{2k-2}}\{Y_{i_{2k-1}}, X_{i_{2k}}\}) \tag{30}$$

A simple observation is that the action

$$([X^i, X^j] - [Y^i, Y^j])^2 \tag{31}$$

also has (all) the odd fuzzy spheres as a trivial solution, since both equations of motion are proportional to the constraint $[X^i, X^j] - [Y^i, Y^j] = 0$.

More realistically, adding

$$\alpha([X^i, X^j] - [Y^i, Y^j])^2 \tag{32}$$

to the previous action still satisfies the odd fuzzy sphere equations of motion, so there is a one-parameter set of actions with the fuzzy odd spheres as solutions.

Dimensional reduction

For classical spheres, the "diameter" of a sphere (reached when one of the coordinates takes an extreme value- maximum or 0) is a sphere of one less dimension. In fact, for any fixed value of one of the coordinates, we get a lower dimensional sphere. Let's check whether we can obtain the same for fuzzy spheres.

The simplest case is the case of applying the procedure twice, for embedding even spheres into even spheres. For n=2k+1, embedding S_{n-1} into S_{n+1} can be done at the level of the algebra:

$$\begin{aligned}
&\sum_{i=1}^{n} \hat{X}_i^2 = 1 - \hat{X}_{n+1}^2 - \hat{X}_{n+2}^2 \\
&[\hat{X}_i, \hat{X}_j] = \epsilon_{iji_3....i_n i_{n+1} i_{n+2}} \hat{X}^{i_3} ... \hat{X}^{i_n} (\hat{X}^{i_{n+1}} \hat{X}^{i_{n+2}})
\end{aligned} \tag{33}$$

and thus if the S_{n+1} operators $[\hat{X}^{i_{n+1}}, \hat{X}^{i_{n+2}}]$, $\hat{X}^2_{i_{n+1}}$, $\hat{X}^2_{i_{n+2}}$ have given eigenvalues, the S_{n+1} algebra reduces to the S_{n-1} algebra.

Let us now try to embed S^3 into S^4, not the explicit representation, but using only the algebra. As we saw, for the even dimensional spheres, we can represent the 5d operators $\hat{X}_\mu$ by Γ_μ. In particular, for dimensional reduction, we can represent $\hat{X}_5 = \Gamma_5$ (more precisely, $\hat{X}_5 = \sum_r \rho_r(\Gamma_5)$).

Since we have

$$[\Gamma_i, \Gamma_j] = \epsilon_{ijkl5} \Gamma_k \Gamma_l \Gamma_5 = \frac{1}{2} \epsilon_{ijkl} \{\Gamma_k, \Gamma_l \Gamma_5\} \tag{34}$$

we see that a good way to set up the dimensional reduction of the fuzzy 4-sphere algebra is

$$[\hat{X}_i, \hat{X}_j] = \frac{1}{2} \epsilon_{ijkl} \{\hat{X}_k, \hat{X}_l \hat{X}_5\} \tag{35}$$

Next, notice that if we dimensionally reduced as follows: $\hat{X}_i = X_i, \hat{X}_i \hat{X}_5 = Y_i$ as one would think natural, then we would get $Y_{ij} = 0, X_{ij} = 2[X_i, X_j]$, which is not what we want (we need to have two independent sets of variables).

Fortunately, we can see from the explicit gamma matrix representation that this is not quite so. In fact,

$$X_i = X_i^+ + X_i^- = \mathcal{P}_- \sum_r \rho_r(\Gamma_i P_+) \mathcal{P}_+ + \mathcal{P}_+ \sum_r \rho_r(\Gamma_i P_-) \mathcal{P}_- \tag{36}$$

where $P_\pm = (1 \pm \Gamma_5)/2$ are the projectors onto given chiralities. That still means that $X_i^\pm \hat{X}_5 = \pm X_i^\pm$, so that $X_i \hat{X}_5 = Y_i = -\hat{X}_5 X_i$. For the dimensional reduction ansatz, we would need to write something like

$$X_i = \mathcal{P}_- \hat{X}_i \frac{1+\hat{X}_5}{2} \mathcal{P}_+ + \mathcal{P}_+ \hat{X}_i \frac{1-\hat{X}_5}{2} \mathcal{P}_- \tag{37}$$

and now (35) will not be true without hats anymore. Rather, without hats, the two sides of (35) will be independent. This form of the dimensional reduction is not very appealing, as we still need the projectors $\mathcal{P}_\pm$ which refer to the explicit representation of the X's as gamma matrices, but we can find no better way of doing it.

In any case, then one has to replace this definition in

$$[\hat{J}_{ij}, \hat{X}_j] = 6\hat{X}_i \tag{38}$$

(the 6 instead of 8 is because the summation is restricted: we don't sum over the 5th coordinate) and using exactly the same calculation as was done with the gamma matrices (except that now we don't use that notation), obtain that it dimensionally reduces to the desired

$$[J_{ij}, X_j] = 6X_i; \quad J_{ij} = A[X_i, X_j] + B\epsilon_{ijkl}\{X_k, Y_l\} \tag{39}$$

where of course $Y_i = X_i \hat{X}_5$.

For the reverse dimensional reduction, of odd sphere to even sphere, e.g. S_3 to S_2, we again start from the dimensional reduction of the representation, to gain insight about the dimensional reduction of the algebra. We have

$$\hat{X}_\mu = \mathcal{P}_{\mathcal{R}_n} \sum_r \rho_r(\hat{\Gamma}_\mu) \mathcal{P}_{\mathcal{R}_n}, \quad \mu = 1, 4 \tag{40}$$

with the dimensional reduction of the gamma matrices (easily generalizable to any odd sphere)

$$\hat{\Gamma}_i = \Gamma_i \otimes \sigma_1; \quad \hat{\Gamma}_4 = 1 \otimes \sigma_2; \quad \hat{\Gamma}_5 = 1 \otimes \sigma_3 \tag{41}$$

Then put

$$\hat{Y}_4 = \sum_r \rho_r(\hat{\Gamma}_4 \hat{\Gamma}_5) = \sum_r \rho_r(1 \otimes \sigma_1) \Rightarrow \hat{X}_i \hat{Y}_4 = \sum_r \rho_r(\Gamma_i) \equiv X_i \tag{42}$$

With this dimensional reduction ansatz, we can write down the dimensional reduction of the SO(4) rotation generator,

$$\hat{J}_{ij} = a[\hat{X}_i, \hat{X}_j] + b\epsilon_{ijk4}\{\hat{X}_k, \hat{Y}_4\} = a[\hat{X}_i, \hat{X}_j] + b\epsilon_{ijk} X_k \tag{43}$$

and thus deduce the dimensional reduction of the equations of motion as follows. If we sandwich $[\hat{J}_{ij}, \hat{X}_j] = 4\hat{X}_i$ between two $\hat{Y}_4$'s we get

$$[J_{ij}, X_j] = 4X_i; \quad J_{ij} = a[X_i, X_j] + b\epsilon_{ijk} X_k \tag{44}$$

It seems though that one still needs to impose the more restrictive equation of motion $[X_i, X_j] = \epsilon_{ijk} X_k$ to get the correct dimensional reduction.

PP wave Matrix model and fuzzy 5-sphere

We expect the fuzzy 5-sphere to be a solution to of the pp wave Matrix Model in [2].

In the supergravity description of M theory, there are two types of giant gravitons in the pp wave background. There are two-spheres that correspond to giant gravitons in AdS_4 or S_4 before the Penrose limit, and whose radius is

$$r = \frac{\pi}{6}\mu p^+ = \frac{\pi}{6}\frac{\mu N}{R} \tag{45}$$

(the second line corresponds to the naive Matrix theory DLCQ) and there are also 5-spheres, that correspond to giant gravitons in AdS_7 or S_7 before the Penrose limit, whose radius obeys

$$r^4 = \frac{8\pi^2}{3}\mu p^+ = \frac{8\pi^2}{3}\frac{\mu N}{R} \tag{46}$$

But in the Matrix model in [2] there are exactly two classical vacuum solutions, a fuzzy two-sphere,

$$[\phi^i, \phi^j] = i\frac{\mu}{6R}\epsilon_{ijk}\phi^k; \quad r = 2\pi\sqrt{\frac{Tr[\sum_i \phi^{i2}]}{N}} \tag{47}$$

and the vacuum, $\phi^i = 0$. The fuzzy two-sphere solution of the Matrix model has the correct (supergravity) radius, as expected.

The Matrix model is written with its coupling $1/g^2$ in front ($g = (R/(\mu N))^{3/2}$), and then we have for the two-sphere in the rescaled variables $\hat{\phi} \sim \mu/g$ (so the solution is classical, $g\hat{\phi} \sim o(1)$, as we have checked). But for the 5-sphere giant graviton, we would obtain $\hat{\phi}^4 \sim \mu/g$, which therefore is quantum in nature in the Matrix model. So the only candidate, the $\phi = 0$ vacuum should quantum mechanically be blown up to a 5-sphere, which then should get the correct radius.

Indeed, [12] have shown that the linear fluctuation spectrum of a 5-brane matches (exactly) with the protected spectrum of excited states about the X=0 vacuum of the Matrix model (found by computing small QM fluctuations and symmetry arguments).

So we would want to obtain the fuzzy 5-sphere as a solution of the exact quantum-mechanically corrected Matrix model.

There are two ways in which it seems possible to do this using our description of the fuzzy 5-sphere. One would be to embed both X_i and Y_i into the ϕ_i of the Matrix model. But we can easily check that this is not possible for the action in (27). We could add

$$\alpha([X^i, X^j] - [Y^i, Y^j])^2 \tag{48}$$

to the action, obtaining (for $\alpha = a$)

$$\begin{aligned} L &= Tr\{\frac{a}{2}([X_i, X_j]^2 - [X_i, X_j][\tilde{Y}_i, \tilde{Y}_j]) - 5(X_i^2 - \tilde{Y}_i^2) \\ &\quad + i\frac{b}{2}\epsilon_{ijklmn}(X_i X_j X_k X_l\{X_m, \tilde{Y}_n\} + \tilde{Y}_i\tilde{Y}_j X_k X_l\{\tilde{Y}_m, X_n\})\} \end{aligned} \tag{49}$$

But it is not clear how we would go about embedding this either.

We should note here that whenever we write a fuzzy sphere action as a Matrix action we implicitly assume that the coupling has been extracted as an overall $1/g^2$, so that we can get back the coupling dependence by rescaling fields and coordinates.

The only other way (without adding the extra term to the action) seems to be to have Y^i being a bilinear in the fermion fields, of the type $\Psi\Gamma^i\Psi$.

This is not so unusual. For instance, the QCD chiral symmetry breaking phase transition order parameter is believed to be $<\bar{q}q>$ (the quark condensate). Also, in the case of the Seiberg [13] and Seiberg-Witten [14] analysis of the $\mathcal{N}=1$ and $\mathcal{N}=2$ susy gauge theories, the order parameter of the chiral symmetry phase breaking is the "meson" field $M=\tilde{Q}Q$, that acquires a nonzero VEV. In both cases, the nonzero VEV of the bilinear in fields appears nonperturbatively, and is essential to the physics.

Therefore it is not unlikely to have a quantum theory with an "effective potential" for the "meson" field $\Psi\Gamma^i\Psi$, of the needed form. It is unfortunately not clear how to calculate it. The only difference from the meson case is that now we want to have an object with gauge indices (not a gauge invariant combination), but given that fields are in the adjoint representation of the gauge group, it is the natural thing to do.

There is a simple way to argue for the presence of the fuzzy 5-sphere action. Myers [15] derived a term $F_{tijk}Tr(X^iX^jX^k)$ that boosted gave $F_{+ijk}Tr(X^iX^jX^k)=\epsilon_{ijk}Tr(X^iX^jX^k)$ for the pp wave (M)atrix model. It came from the classical DBI coupling $\int i_X(C_{(3)})$.

If we would have a dual $C_{(6)}$ field in the same background, it should similarly provide a term

$$F_{+ijklmn}Tr(X^iX^jX^kX^lX^mX^n)=\epsilon_{ijklmn}Tr(X^iX^jX^kX^lX^mX^n) \tag{50}$$

Notice then that a nonlinear susy transformation

$$\begin{aligned}\delta X_n &= \Psi^T\Gamma_n\epsilon\\ \delta\Psi &= \epsilon\end{aligned} \tag{51}$$

would relate it to a term of the desired form

$$\epsilon_{ijklmn}Tr(X^iX^jX^kX^l\{X^m,\Psi^T\Gamma^n\Psi\}) \tag{52}$$

Then this term, together with the usual commutator and mass terms,

$$[X^i,X^j]^2-m^2(X^i)^2 \tag{53}$$

provide half of the terms in the fuzzy 5-sphere action, and the other half are there to cancel the energy when the constraints are satisfied. As the action is valid only on the constraints anyway, this seems enough to argue for the plausibility of the action.

So we have found that the action (27) is a possible candidate for describing the quantum 5-sphere solution of the pp wave Matrix model. This action is unique if we impose the equations of motion and the condition of minimal corrections to the Matrix model, so this gives it a certain degree of plausibility.

We might be worried that the action would have negative energy configurations, but we have to remember that the action is only valid if the constraints are satisfied, and then the energy is automatically zero.

We observe that the action (27) has also $X = Y = 0$ as a solution (along with many others), and thus why would we call the fuzzy 5-sphere solution a blow-up of the X=0 solution of the BMN Matrix model? We only gave an intuitive argument for half of the terms in the action, the rest were guessed by using supersymmetry and the constraints. In reality, there should probably be more terms for $Y_i = \Psi^T \Gamma^i \Psi$ which would force it to be nonzero, and for a solution satisfying our constraints that would mean the X's would be nonzero too.

Note that [16] has also proposed an action that would have the fuzzy 3-sphere as a solution, different than the one proposed in this paper, but that construction is slightly different in scope, being motivated by a 3-brane discretization, and the fuzzy 3-sphere construction used there is different, using a constant matrix depending on the representation.

Finally, we should note that the fuzzy S^4 construction does not cover all possible 4-brane charges, and one needs something else (maybe nonassociativity) to describe the general case. Similarly, it is possible that the fuzzy 5-sphere construction will only provide certain cases for the expected giant 5-sphere graviton.

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